Appendix A

Proof of Proposition 2

Proposition 2. *The extended greedy approach provides a $(1-\frac{1}{e})$-approximation to a WMC with a negative weight.*

Proof. Let $\mathbf{s}^*$ be the optimal solution of the WMC. Let $\mathbf{s}^0$ and $\mathbf{s}^\infty$ be the *optimal* solutions of WMC^0 and WMC^∞ of the extended greedy approach, respectively. Let $\mathbf{c}^*{=}\Phi(\mathbf{s}^*)$, $\mathbf{c}^0{=}\Phi(\mathbf{s}^0)$, and $\mathbf{c}^\infty{=}\Phi(\mathbf{s}^\infty)$. Let the target with negative weight be target i^+.

We first prove the following two claims.

Claim 1. *If $c^*_{i^+}=0$, i.e., target i^+ is not covered by the optimal solution, then $\mathbf{s}^\infty$ is also optimal to the WMC.*

Proof of Claim 1. Since w_{i^+} is replaced with $-\infty$ in WMC^∞, target i^+ must not be covered by the optimal solution of WMC^∞. Thus $c^\infty_{i^+}=0$. We have $\sum_i w_i\cdot c^*_i=\sum_{i\neq i^+} w_i\cdot c^*_i$ and $\sum_i w_i\cdot c^\infty_i=\sum_{i\neq i^+} w_i\cdot c^\infty_i$. Suppose that $\mathbf{s}^\infty$ is not optimal to the WMC. We have $\sum_i w_i\cdot c^*_i>\sum_i w_i\cdot c^\infty_i$, and

$$\sum_{i\neq i^+} w_i\cdot c^*_i+(-\infty)\cdot c^*_{i^+}=\sum_i w_i{\cdot}c^*_i>\sum_i w_i{\cdot}c^\infty_i=\sum_{i\neq i^+} w_i\cdot c^\infty_i+(-\infty)\cdot c^\infty_{i^+}.$$

This means that $\mathbf{s}^*$ is a better solution to WMC^∞ than $\mathbf{s}^\infty$, which is a contradiction. □

Claim 2. *If $c^*_{i^+}=1$, i.e., target i^+ is covered by the optimal solution, then $\mathbf{s}^0$ is also optimal to the WMC when $w_{i^+}<0$.*

Proof of Claim 2. Suppose $\mathbf{s}^0$ is not optimal to the WMC. We have $\sum_i w_i\cdot c^*_i>\sum_i w_i\cdot c^0_i$. Thus,

$$\begin{aligned}&\sum_{i\neq i^+} w_i\cdot c^*_i+w_{i^+}\cdot c^*_{i^+}>\sum_{i\neq i^+} w_i\cdot c^0_i+w_{i^+}\cdot c^0_{i^+}\\ \Rightarrow&\sum_{i\neq i^+} w_i\cdot c^*_i>\sum_{i\neq i^+} w_i\cdot c^0_i-w_{i^+}\cdot(c^*_{i^+}-c^0_{i^+})\\ \Rightarrow&\sum_{i\neq i^+} w_i\cdot c^*_i>\sum_{i\neq i^+} w_i\cdot c^0_i\end{aligned}$$

The last inequality holds since $w_{i^+}\cdot(c^*_{i^+}-c^0_{i^+})\le 0$, and it indicates that $\mathbf{s}^*$ is a better solution to WMC^0 than $\mathbf{s}^0$, which is a contradiction. □

Therefore, if $c^*_{i^+}=0$, the optimal solution of WMC^∞ is optimal to the WMC, and obviously, the objective values induced by this solution are the same for these two problems since the two problems only differ in the weight w_{i^+}, which is however not counted in the solution. Therefore, the greedy algorithm provides a $(1-\frac{1}{e})$ approximation to WMC^∞ (as all weights in WMC^∞ is non-negative) and furthermore the WMC, i.e.,

$$\begin{aligned}&GRD\ge(1-\frac{1}{e})\cdot OPT\\ \Rightarrow&GRD+|w_{i^+}|>(1-\frac{1}{e})\cdot(OPT+|w_{i^+}|)\\ \Rightarrow&\frac{GRD+|w_{i^+}|}{OPT+|w_{i^+}|}>1-\frac{1}{e}.\end{aligned}$$

On the other hand, if $c^*_{i^+}=1$, the optimal solution of WMC^0 is optimal to the WMC. In this case, the optimal objective value of WMC^0 is $|w_{i^+}|$ larger than that of the WMC, i.e., $OPT+|w_{i^+}|=OPT_0$. For the greedy solution of WMC^0, since target i^+ can be either covered or uncovered, we have $GRD=GRD_0-|w_{i^+}|$ or $GRD=GRD_0$, and in both cases $GRD+|w_{i^+}|\ge GRD_0$. Since GRD_0 provides a $(1-\frac{1}{e})$-approximation to OPT_0 (as all weights in WMC^0 is non-negative), we have $GRD_0\ge(1-\frac{1}{e})\cdot OPT_0$. It follows that

$$\begin{aligned}&GRD+|w_{i^+}|\ge(1-\frac{1}{e})\cdot(OPT+|w_{i^+}|)\\ \Rightarrow&\frac{GRD+|w_{i^+}|}{OPT+|w_{i^+}|}\ge 1-\frac{1}{e}.\end{aligned}$$

Therefore, we have in both cases $\frac{GRD+|w_{i^+}|}{OPT+|w_{i^+}|}\ge 1-\frac{1}{e}$, i.e., a $(1-\frac{1}{e})$-approximation. □

Proof of Proposition 3

Proposition 3. *Let the polytope defined by Eq.* (16) *be $\bar{\mathcal{C}}$, and let the polytope of $\bar{\mathbf{c}}$ defined by Eqs.* (17)–(20) *be $\bar{\mathcal{C}}'$. Then $\bar{\mathcal{C}}\subseteq\bar{\mathcal{C}}'$.*

Proof. Suppose $\bar{\mathbf{c}}$ is a vector in $\bar{\mathcal{C}}$. There must be a mixed strategy $\mathbf{x}$ such that $\bar{\mathbf{c}}=\sum_{\mathbf{s}\in\mathcal{S}} x_{\mathbf{s}}\cdot\Phi(\mathbf{s})$. Let $\bar{\mathbf{s}}=\sum_{\mathbf{s}\in\mathcal{S}} x_{\mathbf{s}}\cdot\mathbf{s}$. It follows that $\bar{\mathbf{c}}$ and $\bar{\mathbf{s}}$ satisfies Eqs. (17)–(20) since $\Phi(\mathbf{s})$ and $\mathbf{s}$ satisfy Eqs. (10)–(13). □

Appendix B

A $\frac{1}{K}$-approximation of a Naïve Greedy Approach

As mentioned in the paper, a naïve greedy approach to deal with a WMC with a negative weight is to keep track of the total weight in each step of the main loop (Lines 2–4, Algorithm 1), and choose the maximum record as the final solution. This approach provides a $\frac{1}{K}$-approximation to the WMC. The approach is depicted with Algorithm B-1 below, and the approximation ratio is demonstrated in Proposition B-1.

Algorithm B-1: A naïve greedy approach to deal with a WMC with negative weights

Input: An adjacency matrix $\mathbf{A} = \langle a_{ij} \rangle$
A set of weights $\mathbf{w} = \langle w_i \rangle$
Output: A pure strategy $\mathbf{s}^*$

```
1 r ← 0, s ← 0, c ← 0;
2 r* ← 0, s* ← 0;
3 for k = 1 to K do
4     î ← arg max_{i|s_i=0} Σ_{j | a_ij=1 and c_j=0} w_j;
5     r ← r + Σ_{j | a_îj=1 and c_j=0} w_j;
6     s_î ← 1, c ← Φ(s);
7     if r > r* then r* ← r, s ← s*;
```

Proposition B-1. *Algorithm B-1 provides a $\frac{1}{K}$-approximation to a WMC with a negative weight $w_{i^+} < 0$, and $\frac{1}{K}$ is a tight bound.*

Proof. Let OPT denote the objective value of the optimal solution. Let GRD denote the objective value of of the solution returned by Algorithm B-1. Let x_l denote the total weight of new targets covered by Algorithm B-1 with the l^{th} set that it picks for all $l = 1, \ldots, K$. Note that the solution of Algorithm B-1 may contain less than K sets, say K' sets. Without loss of generality, we let $x_l = 0$ for all $l > K'$. Let $y_l = \sum_{j=1}^{l} x_j$, i.e., the total weights covered by Algorithm B-1 with the first l sets it picks, and let $z_l = OPT - y_l$. We also let $y_0 = 0$ and $z_0 = OPT$.

Claim 1. $x_{l+1} + |w_{i^+}| \geq \frac{z_l + |w_{i^+}|}{K}$.

Proof of Claim 1. Suppose the optimal solution chooses $k \leq K$ sets $S^1, \ldots, S^k$. Let S_l^j denote the elements in S^j that is yet not covered when Algorithm B-1 has picked l sets. Define $w(S) = \sum_{i \in S} w_i$, i.e., the total weight of targets covered by any set S.

a) If i^+ is not in any of $S_l^1, \ldots, S_l^k$, we have $w(S_l^j) \geq 0$ for all $j = 1, \ldots, k$, and $\sum_{j=1}^{k} w(S_l^j) \geq w(\bigcup_{j=1}^{k} S_l^j) = z_l$. Thus there must be one set in $S_l^1, \ldots, S_l^k$, say S_l^j, such that $w(S_l^j) \geq \frac{z_l}{k} \geq \frac{z_l}{K}$, and furthermore $w(S_l^j) + |w_{i^+}| \geq \frac{z_l + |w_{i^+}|}{K}$.

b) If i^+ is in some of $S_l^1, \ldots, S_l^k$, we replace the negative weight with 0 and thus have $\sum_{j=1}^{k} \left(w(S_l^j) + |w_{i^+}| \right) \geq w(\bigcup_{j=1}^{k} S_l^j) + |w_{i^+}| = z_l + |w_{i^+}|$. Similarly, there must be some S_l^j, such that $w(S_l^j) + |w_{i^+}| \geq \frac{z_l + |w_{i^+}|}{k} \geq \frac{z_l + |w_{i^+}|}{K}$.

Since in the $(l+1)^{\text{th}}$ pick Algorithm B-1 picks the set with largest total weight of uncovered new targets, we have $x_{l+1} \geq w(S_l^j)$, $\forall j = 1, \ldots, k$. This is also true if the solution of Algorithm B-1 picks less than K sets, since when the total weight cannot be increased by selecting more sets, the total weight of uncovered targets in each set must be no larger than 0. We conclude that $x_{l+1} + |w_{i^+}| \geq \frac{z_l + |w_{i^+}|}{K}$. □

Claim 2. $z_l - (K-1) \cdot |w_{i^+}| \leq \left(1 - \frac{1}{K}\right)^l \cdot \left(OPT - (K-1) \cdot |w_{i^+}|\right)$.

Proof of Claim 2. According to the definition of x_l and z_l, we have

$$
\begin{aligned}
z_l &\leq z_{l-1} - x_l \\
&\leq \left(1 - \frac{1}{K}\right) \cdot \left(z_{l-1} + |w_{i^+}|\right) \quad \text{(using Claim 1)} \\
\Rightarrow z_l - (K-1) \cdot |w_{i^+}| &\leq \left(1 - \frac{1}{K}\right) \cdot \left(z_{l-1} - (K-1) \cdot |w_{i^+}|\right) \\
&\leq \left(1 - \frac{1}{K}\right)^l \cdot \left(z_0 - (K-1) \cdot |w_{i^+}|\right) \\
&= \left(1 - \frac{1}{K}\right)^l \cdot \left(OPT - (K-1) \cdot |w_{i^+}|\right)
\end{aligned}
$$

□

According to Claim 2, we have

$$
\begin{aligned}
z_K - (K-1) \cdot |w_{i^+}| &\leq \left(1 - \frac{1}{K}\right)^K \cdot \left(OPT - (K-1) \cdot |w_{i^+}|\right) \\
&\leq \frac{1}{e} \cdot \left(OPT - (K-1) \cdot |w_{i^+}|\right),
\end{aligned}
$$

$$
\Rightarrow GRD = OPT - z_K \geq \left(1 - \frac{1}{e}\right) \cdot \left(OPT - (K-1) \cdot |w_{i^+}|\right)
$$

$$
\Rightarrow OPT \leq \frac{e}{e-1} \cdot GRD + (K-1) \cdot |w_{i^+}|.
$$

Therefore,

$$
\begin{aligned}
\frac{GRD + |w_{i^+}|}{OPT + |w_{i^+}|} &\geq \frac{GRD + |w_{i^+}|}{\frac{e}{e-1} \cdot GRD + (K-1) \cdot |w_{i^+}| + |w_{i^+}|} \\
&= \frac{1}{K} \cdot \frac{GRD + |w_{i^+}|}{\frac{e}{K \cdot (e-1)} \cdot GRD + |w_{i^+}|} \geq \frac{1}{K}.
\end{aligned}
$$

Note that the last inequality holds since $GRD \geq 0$ (the initial solution of a zero vector already provides a solution with an objective value of 0, Line 2, Algorithm B-1), and $\frac{e}{K \cdot (e-1)} > 1$.

The Bound is Tight To show that $\frac{1}{K}$ is a tight bound, we construct the following example. Let $w_1 = w_2 = \ldots w_K = W > 0$, $w_{K+1} = -W$, and $w_i = \varepsilon > 0$ for all the other targets. For the adjacency matrix, let $a_{ii} = 1$ for all $i \in [N]$; $a_{i,K+1} = 1$ for all $i = 1, \ldots, K$; and $a_{ij} = 0$ for all other $i, j \in [N]$. Also let $N > 2K + 1$ and $\varepsilon \ll W$. Obviously, the optimal solution is to assign the K resources to target 1 to K, by which a total weight of $(K-1)\cdot W$ is obtained; whereas Algorithm B-1 will pick K targets with weight ε, by witch a total weight of $K\cdot\varepsilon$ is obtained. Thus $\frac{GRD+|w_{i+}|}{OPT+|w_{i+}|} = \frac{K\cdot\varepsilon+W}{(K-1)\cdot W+W} = \frac{1}{K}$.

We conclude that Algorithm B-1 provides a tight $\frac{1}{K}$ approximation to the WMC when $w_{i+} < 0$.

Appendix C

Generalizations of SPEs

In some real-world scenarios, there can be more complex SPEs where the status of a target is more than simply "covered" or "uncovered". In these cases, entries of the adjacency matrix are allowed to take continuous values between 0 and 1, i.e., $\mathbf{A} \in [0,1]^{n\times n}$. This modification reveals the fact that some factors (such as distance) make a difference on how well a target can be protected by a resource. The way a resource allocation determines the protection status, i.e., the way $\Phi(\mathbf{s})$ is defined, leads to the following two generalizations of SPEs.

• **SPE-add**, which assumes that protections of security resources accumulate, i.e., $\Phi(\mathbf{s}) = \big(\phi_1(\mathbf{s}), \ldots, \phi_N(\mathbf{s})\big)$, and

$$\phi_i(\mathbf{s}) = \begin{cases} \sum_j s_j \cdot a_{ji}, & \text{if } \sum_j s_j \cdot a_{ji} < 1 \\ 1, & \text{if } \sum_j s_j \cdot a_{ji} >= 1 \end{cases}. \tag{C-1}$$

• **SPE-max**, which uses the best protection offered by security resources, i.e., $\Phi(\mathbf{s}) = \big(\phi_1(\mathbf{s}), \ldots, \phi_N(\mathbf{s})\big)$, and

$$\phi_i(\mathbf{s}) = \max_j s_j \cdot a_{ji}. \tag{C-2}$$

Note that SPE can be treated as a special case of both SPE-add and SPE-max. We adopt similar column generation approach to these two cases, where only the slave problems need to be reformulated. The *u*-LP for SPEs also provides an upper bound for the *t*-LPs of SPE-add and SPE-max as Eqs. (17)–(20) defines a superset of the feasible marginal coverage spaces for the same reason as stated in Proposition 3.

Slave Problem Formulations

SPE-add: We use an auxiliary variable θ_i to indicate whether $\sum_j s_j \cdot a_{ji} \geq 1$. The slave problem of SPE-add is formulated as the following MILP.

$$\max_{\mathbf{c},\mathbf{s},\langle\theta_i\rangle} \quad w + \sum_i w_i \cdot c_i \tag{C-3}$$

$$\text{s.t.} \quad \mathbf{c} \in [0,1]^N,\ \mathbf{s} \in \{0,1\}^N,\ \langle\theta_i\rangle \in \{0,1\}^N \tag{C-4}$$

$$\sum_i s_i \leq K \tag{C-5}$$

$$-K \cdot \theta_i \leq c_i - \sum_j s_j \cdot a_{ji} \leq 0,\ \forall i \in [N] \tag{C-6}$$

$$\theta_i \leq c_i \leq 1,\ \forall i \in [N] \tag{C-7}$$

When $\sum_j s_j \cdot a_{ji} < 1$, it must be that $\theta_i = 0$ (since otherwise Eq. (C-7) will indicate $c_i = 1$, which contradicts the second ineuqality of Eq. (C-6)), and it follows that $c_i = \sum_j s_j \cdot a_{ji}$; similarly, when $\sum_j s_j \cdot a_{ji} \geq 1$, we have $c_i = 1$.

SPE-max: Similarly, we introduce auxiliary variables $\langle\theta_{ij}\rangle\ \forall i, j \in [N]$, and formulate the slave problem of SPE-max as the following MILP.

$$\max_{\mathbf{c},\mathbf{s},\langle\theta_{ij}\rangle} \quad w + \sum_i w_i \cdot c_i \tag{C-8}$$

$$\text{s.t.} \quad \mathbf{c} \in [0,1]^N,\ \mathbf{s} \in \{0,1\}^N,\ \langle\theta_{ij}\rangle \in \{0,1\}^{N\times N} \tag{C-9}$$

$$\sum_i s_i \leq K \tag{C-10}$$

$$0 \leq c_i - s_j \cdot a_{ji} \leq 1 - \theta_{ij},\ \forall i, j \in [N] \tag{C-11}$$

$$\sum_j \theta_{ij} = 1,\ \forall i \in [N] \tag{C-12}$$

Eq. (C-12) requires that for each $i \in [N]$, one and only one θ_{ij} equals 1. According to Eq. (C-11), θ_{ij} equals 1 only when j maximizes $s_j \cdot a_{ji}$, and $c_i = \max_j s_j \cdot a_{ji}$ is thus satisfied.